CLASS24

Excellent Mathematics–4
Target: NMTC Final Round
Junior (Class-IX/X)

CONTENTS

CLASS24

EXCELLENT MATHEMATICS-4

Target : NMTC Final Round

CHAPTER 1 Algebra

1. Find x,y,z satisfying the equations

$(x + y)(x + y + z) = 66$

$(y + z)(x + y + z) = 99$

$(z + x)(x + y + z) = 77$

2. Let $N_k = 131313...131$ be the $(2k + 1)$-digit number in base 10 formed by k pieces of 13 and appended by 1 at the end. Prove that N_k is not divisible by 31 for any value of k = 1, 2,3 ...

3. We have 12 rods each of 13 units length. They are to be put into pieces measuring 3,4,5 units so that the resulting pieces can be assembled into 13 triangles of sides 3,4,5 units. How should the rods be cut?

4. a,b,c are positive integers satisfying the equations, $5a + 5b + 2ab = 92$, $5b + 5c + 2bc = 136$, $5c + 5a + 2ca = 244$. Find $7a + 8b + 9c$.

5. a,b,c,d are positive integers such that $a^5 = b^4$, $c^3 = d^2$ and $c - a = 19$. Find $d - b$.

6. The pages of a book are numbered 1 through n. When the page numbers of the book were added, one of the page numbes was mistakenly added twice resulting in the incorrect sum 1998. What was the number of the page that was added twice?

7. Given the 7-element set A = {a, b, c, d, e, f, g,}. Find a collection T of 3-element subsets of A such that each pair of elements from A occurs exactly in one of the subsets of T.

8. For which positive integer values of n, the set {1,2,3,4,.... 4n} can be split into n distinct 4 elements subsets {a,b,c,d} such that $a = \frac{b+c+d}{3}$.

9. Prove that, there are infinitely many triplets (x,y,z) of positive integers such that $x^3 + y^5 = z^7$.

10. Seven points are placed inside a square of side 1. Prove that, at least two of them, are at a distance of not greater than $\sqrt{\frac{13}{6}}$.

11. If p, q, r are the roots of the cubic equation $x^3 - 3px^2 + 3q^2x - r^3 = 0$, prove that $p = q = r$.

12. Find all solutions in integers m, n of the equation $(m-n)^2 = \frac{4mn}{m+n-1}$.

13. If a, b, c, x are real numbers such that $abc \neq 0$ and $\frac{bx+(1-x)c}{a} = \frac{cx+(1-x)a}{b} = \frac{ax+(1-x)b}{c}$ then, prove that, $a + b + c = 0$ or $a = b = c$.

14. Multiply the consecutive positive integers until the product 2.4.6.8... becomes divisible by 2001. Find the largest even integer we use to satisfy this condition.

15. In the year 2000, I will be old as the sum of the digits of my birth year. When was I born?

16. Prove that, if a,b(a > b) are prime numbers, each containing at least 2 digits, then $a^4 - b^4$ is divisible by 240. Also prove that, 240 is the greatest common divisor of all numbers which arise in this way.

17. Let $f(x)=\frac{16^x}{16^x+4}$. Evaluate the sum $\frac{1}{2000}+\frac{2}{2000}+...+\frac{1999}{2000}$

18. Arrange the numbers from 1 to 100 as a sequence such that any 11 terms in it (not necessarily consecutive) do not form an increasing or decreasing sequence.

19. Two students from class X and several students from XI participated in a chess tournament. Each participant played one with every other. In each game, the winner has received 1 point, the loser zero and for a drawn game, both players got $\frac{1}{2}$. The two students from class X together scored 8 points and the scores of all participants in class XI are equal. How many students from class XI participated in the tournament?

20. Find all real solutions of the system of equations: $x + y = 2$; $xy - z^2 = 1$

21. A stranger P visited an island, every inhabitant of which is either a 'Knight' who always tells the truth or a 'Knave' who never tells the truth. He met four inhabitants, A, B, C, D. A said : "Exactly one of us is a knave"; B said : "We are all knaves".

Then P asked C : "Is A a knave"? He got an answer (yes or not) from which it was impossible to deduce the truth about A. Is D a knave?

22. Let a, b, c, d be real numbers such that $a^2 + b^2 + (a - b)^2 = c^2 + d^2 + (c - d)^2$. Prove that $a^4 + b^4 + (a - b)^4 = c^4 + d^4 + (c - d)^4$

23. It is known that 3^{1000} contains 478 digits. Let 'a' be the sum of the digits of 3^{1000}, 'b' the sum of the digits of 'a' and 'c' the sum of the digits of 'b'. Find the value of 'c'.

24. The set S consists of 5 integers. If pairs of distinct elements of S are added, the following 10 sums are obtained. 1967, 1972, 1973, 1974, 1975, 1980, 1983, 1984, 1989, 1991. Find the elements of S.

25. Find all integers m, n such that $2mn - 5m + n = 55$

26. A natural number is good if it can be expressed both as a sum of two consecutive natural numbers and as a sum of three consecutive natural numbers. Show that

(i) 2001 is good but 3001 is not

(ii) The product of two good number is good.

(iii) If the product of two numbers is good, then, at least one of them is good.

27. Find all real numbers a and b such that $x^2 + ax + b^2 = 0$ have at least one common root.

28. A school has 281 boys and girls from seven countries. Suppose among any six students, there are at least two who have the same age. Prove that there are five boys from the same country having the same age or there are five girls from the same country having the same age.

29. Prove that the product of the first 1000 positive even integers differs from the product of the first 1000 positive odd integers by a multiple of 2001.

30. Determine the least positive value taken by the expression $a^3 + b^3 + c^3 - 3abc$ as abc vary over all positive integers. Find also all triples (a,b,c) for which this least value is attained.

31. In the given multiplication, a and b are natural numbers. Find a + b.

$$\begin{array}{r} 3a \\ \times\ b2 \\ \hline 70 \\ 140 \\ \hline 1470 \end{array}$$

32. Find all pairs (x, y) where (x, y) are integers such that $x^3 + 11^3 = y^3$.

33. If the quadratic $ax^2 + bx + c$ takes rational values for more than two rational values of x, then, show that a,b,c are all rational numbers.

34. Show that there are no solutions in integers such that $\frac{14x+5}{9}$ and $\frac{17x-5}{12}$ are both integers.

35. If $a \neq 0$, $b \neq 0$, $c \neq 0$ and if

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{a+x}=0$$

$$\frac{1}{a}+\frac{1}{c}+\frac{1}{a+y}=0$$

$$\frac{1}{a}+\frac{1}{x}+\frac{1}{y}=0$$

Prove that a + b + c = 0.

36. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$ Prove that $\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{1}{a^3+b^3+c^3}$.

37. Let $1 < a_1 < a_2 \ldots < a_{51} < a_{142}$. Prove that among the 50 consecutive differences $(a_i - a_{i-1})$, i = 2,3,4,...51, some value must occur at least 12 times.

38. Prove that in any perfect square, the three digits immediately to the left of the unit digit cannot be 101. (For example .... 101x cannot be a perfect square).

39. Show that there are no integers a, b, c for which $a^2 + b^2 - 8c = 6$

40. Given that $N = 2^n (2^{n+1} - 1)$ and $2^{n+1} - 1$ is a prime number, show that the sum of the divisors of N is 2 N.

41. If $a = 2011^{2010}$, $b = 2011^{2011}$, $c = (2010 + 2011)^{2010 + 2011}$ and d = 2011, find the value of $\frac{bc(a+d)}{(a-b)(a-c)}+\frac{ac(b+d)}{(b-a)(b-c)}+\frac{ab(c+d)}{(c-a)(c-b)}$

42. Let $A = \{a^2 + 4ab + b^2 | a,b$ are positive integers$\}$. Prove that $2015 \neq A$.

43. Find integers x, y, z such that

$x^2z + y^2z + 4xy = 40$

$x^2 + y^2 + xyz = 20$

44. If p, q, r are the roots of the cubic equation $x^3 - 3px^2 + 3qx - r^3 = 0$, then compare p,q,r

45. Determine all non-negative integral pairs (x, y) for which $(xy - 7)^2 = x^2 + y^2$

46. Find all integers x, y, z such that $\frac{2}{x^2}+\frac{3}{y^2}+\frac{4}{z^2}=1$

47. Find integers a, b, c such that $a^2 + b^2 - 8c = 6$.

48. Find the largest integer 'n' such that (n + 10) divides $n^3 + 100$.

49. Find all integers x satisfying the equation $\sqrt{x}+x^{\frac{2}{3}}=15972$.

50. Find the least L.C.M of 20 natural numbers not necessarily different whose sum is 801.

51. In a village 1998 persons volunteered to clean up for a fair, a rectangular field with integer sides and perimeter equal to 3996 feet. For the purpose of the field was divided into 1998 equal parts. If each part had an integer area (measured in square feet) find the length and breadth of the field.

52. Determine the number of points (x, y) on the hyperbola, $2xy - 5x + y = 55$ for which both x, y are integers.

53. If x, y are integers such that $(x - y)^2 + 2y^2 = 27$, what are the possible values of x.

54. Find all pairs of positive integers (x, y) such that $x^3 - y^3 = xy + 61$.

55. Find the g.c.d of the set $\{n^{13} - n \mid n$ is a natural number$\}$.

56. Find all real x, y, z satisfying the equations

$$\frac{4x^2}{1+4x^2} = y,\ \frac{4y^2}{1+4y^2} = z,\ \frac{4z^2}{1+4z^2} = x$$

57. Find all rational numbers a,b,c such that the equation $x^3 + ax^2 + bx + c = 0$ has roots a,b,c.

58. Find integers 'a' and 'b' such that $(x^2 - x - 1)$ divides $ax^{17} + bx^{16} + 1$.

59. If a, b, c, d are the roots of the equation $x^4 - \pi x - \sqrt{1999} = 0$, find the equation whose roots are

$$\frac{a+b+c}{d^2}, \frac{a+b+d}{c^2}, \frac{a+c+d}{b^2}, \frac{b+c+d}{a^2}$$

60. If a, b, c, x are real numbers such that $a + b + c \neq 0$ and $\frac{xb+(1-x)c}{a}=\frac{xc+(1-x)a}{b}=\frac{xa+(1-x)b}{c}$ then find relation between a, b, c.

61. Find value of $\frac{x(y+z)}{(x-y)(x-z)}+\frac{y(z+x)}{(y-z)(y-x)}+\frac{z(x+y)}{(z-x)(z-y)}$

62. Calculate the sum $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{99}{100!}$

63. Find all real x, y satisfying $x^3 + y^3 = 7$ and $x^2 + y^2 + x + y + xy = 4$

64. Find all the real numbers x, y satisfying $x^8 + y^8 = 8xy - 6$

65. Find all cubic polynomials p(x) such that $(x - 1)^2$ is a factor of $p(x) + 2$ and $(x + 1)^2$ is a factor of $p(x) - 2$

66. Find positive integers x,y,z such that x<y<z and $\frac{1}{x}-\frac{1}{xy}-\frac{1}{xyz}=\frac{19}{97}$.

67. (a) If none of a, b, c, x, y, z is zero, and $\frac{x^2(y+z)}{a^3}=\frac{y^2(z+x)}{b^3}=\frac{z^2(z+y)}{c^3}=\frac{xyz}{abc}=1$

Prove that $a^3 + b^3 + c^3 + abc = 0$

(b) Solve for x, y, z : $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=\frac{y}{x}+\frac{z}{y}+\frac{x}{z} = x + y + z = 3$

68. A merchant bought a quantity of cotton; he exchanged this for oil and he sold the oil. He observed that the number of kg of cotton, the number of liters of oil obtained for each kg and the number of rupees for which he sold formed a decreasing geometric progression. He calculated that if he had obtained 1 kg more of cotton, one liter more of oil for each kg and Rs. 1 more for each liter, he would have obtained Rs. 10169 more, whereas if he had obtained one kg less of cotton and one liter less of oil for each kg and Rs. 1 less for each liter, he would have obtained Rs. 9673 less. How much did he actually receive ?

69. There are three towns A, B and C. a person walking from A to B, driving from B to C and riding a horse from C to A completes the journey in $15\frac{1}{2}$ hours. By driving from A to B riding a horse from B to C and walking from C to A, he could make the journey in 12 hours. On foot he could make the journey in 22 hours, on horseback in $8\frac{1}{4}$ hours and driving in 11 hours. To walk 1 KM, ride 1 KM and drive 1 KM, he takes altogether half an hour. Find the rates at which he travels and the distance between the towns.

70. Find all positive integeral solutions x, y, z of the equation $xy + yz + zx = xyz + 2$.

71. (a) Find all positive real number x, y, z which satisfy the following equations simultaneously.

$$x^3 + y^3 + z^3 = x + y + z \quad (1)$$

$$x^2 + y^2 + z^2 = xyz \quad (2)$$

(b) Do there exist 10 distinct integers such that the sum of any 9 of them is a perfect square?

72. (a) a, b, c are distinct real number such that

$$a^3 = 3(b^2 + c^2) - 25$$

$$b^3 = 3(c^2 + a^2) - 25$$

$$c^3 = 3(a^2 + b^2) - 25$$

Find abc.

(b) Let $a = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \ldots + \frac{1}{2015^2}$

Find the largest integer $\leq a$.

73. f(x) is a fifth degree polynomial. Given that $f(x) + 1$ is divisible by $(x - 1)^3$ and $f(x) - 1$ is divisible by $(x + 1)^3$, find f(x).

74. (a) If a, b, c are positive real numbers and $a + b + c = 50$ and $3a + b - c = 70$. If $x = 5a + 4b + 2c$, find the range of values of x.

(b) The sides a, b, c of a triangle ABC satisfy the equation $a^2 + 2b^2 + 2016c^2 - 3ab - 4033bc + 2017ac = 0$

Prove that b is the arithmetic mean of a, c.

75. a, b, c are positive real numbers. Find the minimum value of $\frac{a+3c}{a+2b+c} + \frac{4b}{a+b+2c} - \frac{8c}{a+b+3c}$

76. (a) Find all prime numbers p such that $4p^2 + 1$ and $6p^2 + 1$ are also primes.

(b) Determine real numbers x, y, z, u such that

$$xyz + xy + yz + zx + x + y + z = 7$$

$$yzu + yz + zu + uy + y + z + u = 9$$

$$zux + zu + ux + xz + z + u + x = 9$$

$$uxy + ux + xy + yu + u + x + y = 9$$

77. If x, y, z, p, q, r are distinct real numbers such that

$$\frac{1}{x+p}+\frac{1}{y+p}+\frac{1}{z+p}=\frac{1}{p}$$

$$\frac{1}{x+q}+\frac{1}{y+q}+\frac{1}{z+q}=\frac{1}{q}$$

$$\frac{1}{x+r}+\frac{1}{y+r}+\frac{1}{z+r}=\frac{1}{r}$$

find the numerical value of $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}$.

78. Prove that $x^4 + 3x^3 + 6x^2 + 9x + 12$ can not be expressed as a product of two polynomials of degree 2 with integer coefficients.

79. If a, b, c, d are positive real numbers such that $a^2 + b^2 = c^2 + d^2$ and $a^2 + d^2 - ad = b^2 + c^2 + bc$, find the value of $\frac{ab+cd}{ad+bc}$.

ANSWERS

1. (2,4,5) and (–2,– 4, – 5)

3. 3 rods are to be cut to pieces of 3, 3, 3, 4 units, 4 rods into pieces of 5, 5, 3 units & 5 rods into pieces of 4, 4, 5 units.

4. 172 **5.** $10^3 - 3^5$ **6.** 45

7. {a,b,c}, {a,d,e}, {a,f,g}, {b,d,f}, {b,e,g}, {c,e,f} and {c,d,g} **8.** 8k

12. $\{(m,-m): m \in Z\}$; $\left\{\left(\frac{v(v+1)}{2}\right), \frac{v(v-1)}{2} : v \in Z, |v| \ge 2\right\}$ **14.** 58 **15.** 1981 **17.** $999\frac{1}{2}$

18. One arrangement :

10, 9, 8, 7, 6, 5, 4, 3, 2, 1
20, 19, 18, 17, 16, 15, 14, 13, 12, 11
30, 29...22, 21
⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮
100, 99, 98, 97, 96, 95, 94, 93, 92, 91

OR

Another arrrangement :

91, 92, 93, 94, 95, 96, 97, 98, 99, 100
81, 82, 83, 84, 85, 86, 87, 88, 89, 90
71, 72...79, 80
⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮
1, 2, 3, 4, 5, 6, 7, 8, 9, 10

19. 7 or 14 **20.** x = 1, y = 1, z = 0 **21.** No, he is a knight **23.** 9 **24.** {983, 984, 989, 991, 1000}

25. 16 pairs : (0, 55), (1,20), (2,13), (3,10), (7,6), (7,6), (10,5), (17,4), (52,3), (–53, 2), (–18,1), (–11,0), (–8, –1), (–4, –5), (–3,–8) (–2,–15), (–1,–50)

27. a = b, or for real roots a = b = 0 **30.** 4, $(a,b,c) \in \{(1,1,2), (1,2,1), (2,1,1)\}$ **31.** 9

32. (–11, 0) & (0,11) **41.** d = 2011 **43.** (1, 1, 18), (–1, –1, 18), (2, 2, 3) and (–2, –2, 3)

45. (7,0), (0,7), (3,4), (4,3) **46.** (± 3, ± 2, ± 12) and (± 3, ± 3, ± 3)

47. No such integers a, b, c are posible **48.** n = 890 **49.** 11^6 **51.** Length = 1332 breadth = 666

52. 16 pairs **53.** 0, ± 4, ± 6 **54.** Only one pair (6, 5) **55.** 2730 **56.** $x = \frac{1}{2}, y = \frac{1}{2}, z = \frac{1}{2}$

57. (a, b, c) = (0, 0, 0), (1, –2, 0), (1, –1, –1) **58.** a = 987, b = –1597

59. $\sqrt{1999}\,x^4 + \pi x^3 - 1 = 0$ **60.** a = b = c **61.** –1 **62.** $1 - \frac{1}{100!}$

63. (2, –1) or (–1, 2) **64.** x = y = –1 **65.** $x^3 - 3x$ **66.** x = 5, y = 49 & z = 97

68. Rs. 64,000

70. (x, y, z) ≡ (1, 1, 1), (2, 3, 4), (2, 4, 3), (3, 2, 4), (3, 4, 2), (4, 2, 3), (4, 3, 2)

71. (a) None, (b) 376, 349, 304, 241, 160, 61, –56, –191, –344, –515

72. (a) 2, (b) $1\frac{2014}{2015}$ **73.** $f(x) = \frac{-3}{8}x^5 + \frac{5}{4}x^3 - \frac{15}{3}x$ **74.** (a) 190 < x < 210

75. $12\sqrt{2} - 17$ **76.** (a) p =5, (b) x = y = z = 1; u = 3/2 **77.** 0 **79.** $\frac{\sqrt{3}}{2}$

CHAPTER 2 Geometry

1. Nine lines drawn parallel to the base of a triangle divide the other two sides into 10 equal parts and the area into 10 distinct parts. If the area of the largest of these parts is 1997 sq. cms, find the area of the triangle.

2. The measures of length of the sides of a triangle are integers and that of its area is also an integer. One side is 21 and the perimeter is 48. Find the measure of the shortest side.

3. ABC is an isosceles triangle with $\angle B = \angle C = 78°$. D and E are points on AB, AC respectively such that $\angle BCD = 24°$ and angle CBE = 51°. Find $\angle BED$.

4. In ΔABC, BC = 20, median BE = 18 and median CF = 24 (E, F are midpoints of AC, AB respectively). Find the area of ΔABC.

5. Let ABCD be a convex quadrilateral in which $\angle BAC = 50°$, $\angle CAD = 60°$, $\angle CBD = 30°$, $\angle BDC = 25°$; If E is the the point of intersection of AC and BD, find $\angle AEB$.

6. Points M and N lie inside an equilateral triangle ABC such that $\angle MAB = \angle MBA = 40°$; $\angle NAB = 90°$; $\angle NBA = 30°$. Prove that MN is parallel to BC.

7. ABCD is a square with length of a side 1 cm. An octagon is formed by lines joining the vertices of the square to the midpoints of opposite sides. Find the area of the octagon.

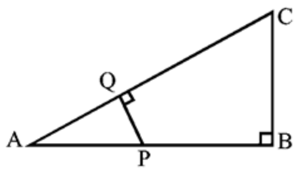
8. Prove that the inradius of a right angled triangle with integer sides is also an integer.

9. Given the rhombus ABCD with $\angle A = 60°$. The points F, H and G are marked on the segments AD, DC and the diagonal AC so that DFGH is a parallelogram. Prove that the triangle FBH is equilateral.

10. Suppose the angle formed by the two rays OX and OY is the acute angle α and A is a given point on the ray OX. Consider all circles touching OX at A and intersecting OY at B, C. Prove that the incentres of all triangles ABC lie on the same st. line.

11. In ΔABC, D is a point on BC such that AD is the internal bisector of $\angle A$. Suppose $\angle B = 2\angle C$ and CD = AB. Prove that $\angle A = 72°$.

12. In the figure, $\angle Q$ and $\angle B$ are right angles. If AQ = 15, BC = 16, AP = 17, find QC.



13. In a right angled triangle, if the square of the hypotenuse is twice the product of the other two sides, prove that the triangle is isosceles.

14. Let ABC be a triangle with AC > BC. Let D be the midpoint of the arc AB that contains C, on the circumcircle of ΔABC. Let E be the foot of the perpendicular from D on AC. Prove that AE = EC + CB.

15. Consider a convex quadrilateral ABCD in which K, L, M, N are the midpoints of the sides AB, BC, CD, DA respective. Suppose

(a) BD bisects KM at Q (b) QA = QB = QC = QD and

(c) $\frac{LK}{LM} = \frac{CD}{CB}$.

Prove that ABCD is a square.

16. A triangle ABC has its circumcentre at O and M is the midpoint of the median through A. If OM is produced to N such that OM = MN, prove that N lies on the altitude through A.

17. Two sides of a triangle are $\sqrt{3}$ cms and $\sqrt{2}$ cms. The medians to these sides are perpendicular to each other. Find the third side.

18. Let ABCD be a convex quadrilateral. P, Q, R, S be the midpoint of AB, BC, CD, DA respectively such that the triangle AQR and CSP are equilateral. Prove that ABCD is a rhombus. Determine its angles.

19. In ΔABC, let D be the midpoint of BC. If $\angle ADB = 45°$ and $\angle ACD = 30°$, determine $\angle BAD$.

20. In ΔABC, $\angle A = 75°$, $\angle B = 60°$, $\angle C = 45°$. Also CF and AD are the altitudes from C and A respectively. If H is the orthocentre and O is the circumcentre, prove that, O is the incentre of ΔCHD.

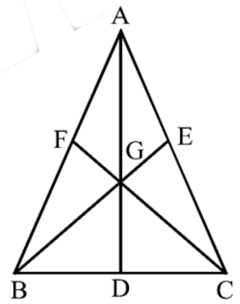
21. Assume that ΔABC is isosceles with $\angle ABC = \angle ACB = 78°$. Let D and E be points on sides AB and AC respectively so that $\angle BCD = 24°$ and $\angle CBE = 51°$. Find $\angle BED$ and justify.

22. ABC and DAC are two isosceles triangles with $\angle BAC = 20°$ and $\angle ADC = 100°$. Show that AB = BC + CD.

23. Let ABCD be a quadrilateral inscribed in a circle. Let M be the point of intersection of the diagonals AC and BD and let E, F, G and H be the feet of the perpendiculars from M on the sides AB, BC, CD, DA respectively. Find the centre of the circle that can be inscribed in the quadrilateral EFGH.

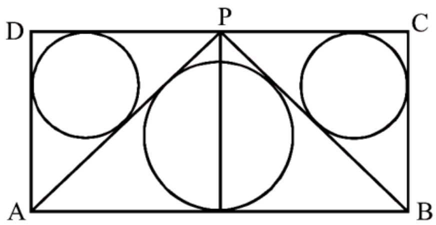
(i.e. touching all its sides)

24. Two sides of a triangle are $\sqrt{3}\,\text{cms}$ and $\sqrt{2}\,\text{cms}$. The medians to these sides are perpendicular to each other. Find the third side.

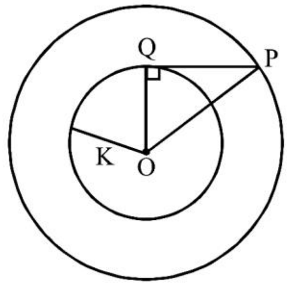


25. Determine all distinct triangles having one side of length 6, with the other two sides being integers, and the perimeter numerically equal to the area.

26. In the adjoning figure ABCD is a rectangle. Triangle PAB is isosceles. The radius of each of the smaller circles is 3 cm and the radius of the bigger circle is 4 cm. Find the length and breadth of the rectangle.

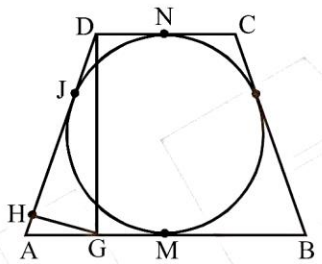


27. 2009 concentric circles are drawn with radii 1 unit to 2009 units. From a point on the outer most circle, tangents are drawn to the inner circles. How many of these tangents are of integer length ?

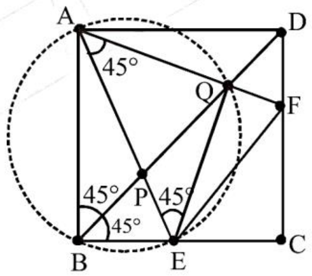


28. The sum of the 3 concurrent edges of a rectangular block is 19cm, its volume is 144 cm^3 and its diagonal is 13 cm. Find the dimensions of the rectangular block.

29. ABCD is an isosceles trapezoid with AB||CD and circumscribed about a circle with CD < AB and BC = AD. DG is drawn perpendicular to AB and GH is drawn perpendicular to DA. Prove that DA,DG and DH are respectively the arithemetic mean, geometric mean and harmonic mean of AB and CD.



30. ABCD is a square. E and F are points respectively on BC and CD such that $\angle EAF = 45°$. AE and AF cut the diagonal BD at P, Q respectively. Then $\frac{\text{Area of } \Delta AEF}{\text{Area of } \Delta APQ} =$



31. A triangle ABC has incentre I. Points X,Y are located on the line segments AB, AC respectively so that BX. AB = IB^2 and CY. AC = IC^2. Given that X,I,Y are collinear, find the possible value of the measure of angle A.

32. Suppose $A_1A_2A_3 .... A_n$ is an n-sided regular polygon such that

$$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$$

Determine n, the number of sides of the polygon.

33. Let P be an interior point of a triangle ABC and let BP and CP meet AC and AB in E and F respectively. If [BPF] = 4, [BPC] = 8 and [CPE] = 13, find [AFPE]. (Here [] denotes the area of a triangle or a quadrilateral as the case may be)

34. The area of a parallelogram is 300 sq cm and its diagonals intersect at 30°. If the sum of the lengths of the diagonals is 80 cms, find the lengths of the sides of the parallelogram.

35. Determine all triples {a, b, c} of positive integers which are the lengths of the sides of a triangle inscribed in a circle of diameter $6\frac{1}{4}$.

36. In triangle ABC, AB = 52, BC = 64, CA = 70 and assume that P and Q are points chosen on AB, AC respectively, such that the triangle APQ and quadrilateral PBCQ have the same area and same perimeter. Find value of PQ^2.

37. In a non-degenerate triangle ABC, $\angle C = 3\angle A$, BC = 27 and AB = 48. Find AC

38. In a triangle ABC,D is a point on BC such that AD is the internal bisector of $\hat{A}$. Suppose B = 2C and CD = AB. Find $\angle A$.

39. In a ΔABC, D is the midpoint of BC. If $A\hat{D}B = 45°$, $A\hat{C}D = 30°$, determine $B\hat{A}D$.

40. If $\angle A$ of a triangle ABC is doubled and the lengths of the sides AB and AC are kept the same, the area of the triangle remains the same. Find $\angle A$.

41. A cyclic octagon ABCDEFGH has sides a,a,a,a,b,b,b,b respectively. Find the radius of the circumcircle of this octagon.

42. A circle passes through the vertex C of a rectangle ABCD and touches its sides AB and AD at M and N respectively. If the distances from C to the line segment MN is equal to 5 units find the area of the rectangle ABCD.

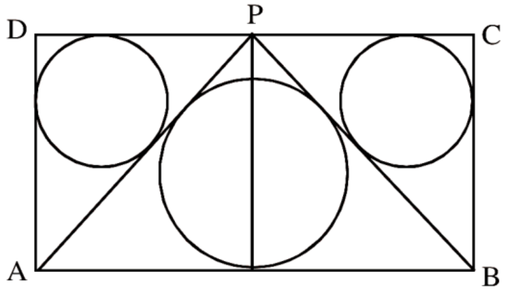
43. A circle of radius r touches a straight line at a point M. Two points A and B are chosen on this line on opposite sides of M such that MA = MB = a. Find the radius of the circle passing through A and B touching the given circle.

44. A rhombus has half the area of the square with the same side-length. Find the ratio of the long diagonal to the short one.

45. Two of the altitudes of a scalene triangle have lengths 4 and 12. If the length of the remaining altitude is also an integer, then its maximum value is

46. ABC is a triangle with side lengths 13, 14 and 15 units. If I is the incentre and R is its circum radius, then the value of $\frac{AI.BI.CI}{R}$ is equal to (in Sq. units)

47. In the adjoining figure ABCD is a rectangle. Triangle PAB is isosceles. The radius of each of the smaller circles is 3 cm and the radius of the bigger circle is 4 cm. Find the length and breadth of the rectangle.



48. In a triangle ABC, BC ≠ CA. The bisectors of $\angle A$ and $\angle B$ meet BC and CA at X and Y respectively; the two bisectors intersect at I. If IX = IY, then find $\angle C$

49. Let D,E,F be the midpoints of the sides BC, CA and AB respectively of triangle ABC. AB = 16,BC = 21 and CA = 19. The circum-circles of the triangles BDF and CDE cut at P other than D. Then $\angle BPC$=

50. ABCD is a rhombus in which B is obtuse. Perpendiculars are drawn from B to the sides and 'a' is the length of each perpendicular. The distance between their feet is 'b'. Area of the rhombus is

51. ABC is a triangle. Perpendiculars BM and CN are drawn to the tangent at A to the circumcircle of ΔABC. The tangent at A meets BC produced at D. If BC = 5 cm, the shorter perpendicular CN = 6cm and AD = $5\sqrt{6}$ cm, the area of the trapezoid BCNM (in cm^2) is

52. ABC is a triangle with AB = 13 cm, BC = 15 cm, and CA = 14 cm. AD and BE are the altitudes from A and B to BC and AC respectively. H is the point of intersection of AD and BE. Then the ratio $\frac{HD}{HB}$ =

53. ABCD is a cyclic quadrilateral in a circle of radius r. AB is a diameter of the circle. CD is parallel to AB. If CD = b, AD = BC = a. The value of $\frac{2r^2 - a^2}{br}$ is

54. PQRS is a common diameter of three circles. The area of the middle circle is the average of the areas of the other two. If PQ = 2, RS =1, find the length of QR.

55. ABCD is a quadrilateral inscribed in a circle of center O. Let BD bisect OC perpendicularly. P is a point on the diagonal AC such that PC = OC. BP cuts AD at E and the circle ABCD at F. Prove that PF is the geometric mean of EF and BF.

56. ABCD is a square E and F are points on BC and CD respectively such that AE cuts the diagonal BD at G and FG is perpendicular to AE. K is a point on FG such that AK = EF. Find the measure of the angle EKF.

57. Two circle S_1 and S_2 intersect at point A and B. The tangent at A to S_1 meets S_2 at C and the tangent at A to S_2 meets S_1 at D. A line through A interior to the angle CAD meets S_1 at M and S_2 at N and meets the circumcircle of triangle ACD at P. Prove that AM = NP.

58. ABC is an equilateral triangle. D is a point inside the triangle such that DA = DB. E is a point that satisfies the two conditions

(i) ∠DBE = ∠DBC and (ii) BE = AB

Find the measure of the ∠DEB.

59. ABC and DBC are two equilateral triangle on the same base BC. A point P is taken on the circle with center D and radius BD. Show that PA, PB, PC are the sides of a right angled triangle.

60. In an isosceles triangle ABC, AB = BC. The bisector AD of ∠A meets the side BC at D. The line perpendicular to AD through D meets AB at F and AC produced at E. Perpendiculars from B and D to AC are respectively BM and DN. If AE = 2016 units, find the length MN.

61. Two circles with centres at P and Q and radii $\sqrt{2}$ and 1 respectively intersect each other at A and D and PQ = 2 units. Chord AC is drawn to the bigger circle to cut it at C and the smaller circle at B such that B is the midpoint of AC. Find the length of AC.

62. ADC and ABC are triangles such that AD = DC and CA = AB. if ∠CAB = 20° and ∠ADC = 100°, without using Trigonometry, prove that AB = BC + CD.

63. In a scalene triangle ABC, ∠BAC = 120°. the bisectors of the angles A, B and C meet the opposite sides in P, Q and R respectively. Prove that the circle on QR as diameter passes through the point P.

64. 2n + 1 segments are marked on a line. Each of these segments intersects at least n other segments. Prove that one of these segments intersects all other segments.

65. Square ABCD and BCFG are drawn outside of a triangle ABC. Prove that if DG is parallel to AC then the triangle ABC is isosceles.

ANSWERS

1. 199700/19 **2.** 10 cm **3.** 12° **4.** 288 **5.** 95° **7.** 1/6

12. 19 **17.** 1 cm **18.** 60°, 120°, 60°, 120° **19.** 30°

23. Centre is a point on the internal bisectors of angles of the quadrilateral, the perpendicular distance of M from the sides are equal

24. 1 cm **25.** (6, 8, 10), (6, 25, 29) **26.** Length = 24, Breadth = 9

27. No integer tangent is possible. **28.** Edges = 3, 4, 12

31. $\angle A = 60°$ **32.** $n = 7$ **33.** 143 **34.** $10\sqrt{10-3\sqrt{3}}$ and $10\sqrt{10+3\sqrt{3}}$

35. (5, 5, 6) **36.** 3255 **37.** 35 **40.** 60° **41.** $r=\sqrt{(a^2+b^2+\sqrt{2}ab)/2}$

42. 25 sq. units **43.** $R=\dfrac{a^2+4r^2}{4r}$ **44.** $2+\sqrt{3}$

45. 8 **46.** 64 **47.** 24, 9 **48.** 60° **49.** $2\angle BAC$

50. $\dfrac{2a^4}{b\sqrt{4a^2-b^2}}$ **51.** 30 **52.** $\dfrac{3}{5}$ **53.** 1 **54.** $\sqrt{6}-1$

56. 135°

58. $\angle DEB = 30°$ **60.** 504 units **61.** $\sqrt{\dfrac{7}{2}}$

CHAPTER 3 Number System

1. a679b is a 5-digit number in decimal system (base ten) which is divisible by 72. Find a and b.
2. Show that there exists no integer n such that the sum of the digits of n^2 is 2000.
3. Show that $4^{1999} + 7^{1999} - 2$ is divisible by 9.
4. Let n be a positive integer greater than 5. Show that, atmost eight numbers of the set $\{n + 1, n + 2 .... n + 30\}$ can be primes.
5. Starting with the four-digit number N in base 10, we subtract a 3-digit number formed by dropping of the last digit (on the right) of N, and then, we add the 2-digit number formed by dropping the last 2 digits of N and add the one-digit number formed by dropping the last three digits of N. What is N if the above computations yield 1999 ?
6. Find all integers n such that $\frac{n^3 - 1}{5}$ is a prime number.
7. Find the number of positive integers which divide 10^{999} but not 10^{998}.
8. Given natural numbers a, b, c such that $a^3 - b^3 - c^3 = 3abc$ and $a^2 = 2(b + c)$, find the values of a, b and c ; (a, b, c need not be all different).
9. The unit of digit of a square number is 6. Find the ten's digit of that number.
10. We define "funny numbers" as follows : Every single digit prime is funny. A prime number with 2 or more digits is funny, if the two numbers obtained by deleting either its leading digit or its unit's digit are both funny. Find all funny numbers.
11. A sports meet was organized for four days. If on each day, half of the existing medals and one more medal was awarded, find the number of medals awarded on each day.
12. Given that $2^n(2^{n+1} - 1)$ and $2^{n+1} - 1$ is a prime number, show that
    (a) sum of the divisors of N is 2N.
    (b) sum of the reciprocals of the divisors of N is 2.
13. A number with 8 digits is a multiple of 73 and also a multiple of 137. Find the second digit from the left ?
14. Let m be the least positive integer such that 1260m is the cube of a natural number.
    Show that $1000 < m < 10000$.
15. If $(43)_x$ in base x number system is equal to $(34)_y$ in base y number system, find the possible value for x + y.
16. A six digit number is said to be lucky if the sum of its first three digits is equal to the sum of its last three digits. Prove that the sum of all six digit lucky numbers is divisible by 13.
17. Show that in the year 1996, no one could claim on his birthday, his age was the sum of the digits of the year in which he was born. Find the last year prior to 1996 which had the same property.
18. Prove that in any perfect square the three digits immediately to the left of the unit digit cannot be 101.
19. The number of students in a university is a perfect square. In one year 2000 more students joined the original strength and the new strength is one more than a perfect square. The next year 2000 more students joined newly and the new strength is again a perfect square. What was the original strength of the university ?
20. Let x, y, z be three positive reals, each less than 4. Prove that atleast one of the numbers $\frac{1}{x}+\frac{1}{4-y}, \frac{1}{y}+\frac{1}{4-z}, \frac{1}{z}+\frac{1}{4-x}$ is greater than or equal to 1.

21. Find all integers n such that $\frac{7n-12}{2^n}+\frac{2n-14}{3^n}+\frac{24n}{6^n}=1$.

22. Prove that any prime no $(2^{2n}+1)$ cannot be represented as a difference of two fifth powers of integers.

23. abc is a three digit number. ab, bc, ca are two digit numbers. Determine all three digit numbers abc such that abc = ab + bc + ca.

24. Three are two natural numbers whose product is 192. It is given that the quotient of the arithmetic mean to the harmonic mean of their greatest common measure and the least common multiple is $\frac{169}{48}$. Find the numbers.

25. Find all the positive integral solution of the equation $\frac{1}{x}+\frac{1}{y}=\frac{1}{2013}$

26. Find the least number whose last digit is 7 and which becomes 5 times larger when this last digit is carried to the beginning of the number.

27. All the 2-digit numbers from 19 to 93 are written consecutively to form the number N = 19202122... 919293. Find the largest power of 3 that divides N.

28. Determine all pairs (m,n) of positive integers for which $2^m + 3^n$ is a square.

29. Find all four-digit numbers having the following properties.

(i) it is a square,

(ii) its first two digit are equal to each other and

(iii) its last two digit are equal to each other

30. Determine the largest 3-digit prime factor of the integer $\binom{2000}{1000}$

31. Find the remainder when 2^{1990} is divided by 1990.

32. Find the greatest common divisor of $2^{2^{1993}}+1$ and $2^{2^{1995}}+1$

33. Find a six digit number that increases 6 times when its last three digits are carried to the beginning of the number without their order being changed.

34. Find the largest possible value of k for which 3^{11} is expressible as the sum of k consecutive positive integers.

35. Find the smallest positive integer whose cube ends with 888.

36. The system of equations (a + 1) (b – 5) = N and (a – 1) (b + 5) = N has integer solutions when N = 1995. Determine the next smallest integer N > 1995 which also yields integer solutions to the given equations.

37. Let $A_1 = 1998^{1998}$ and for n > 1, let A_n denote the sum of the digits of A_{n-1}. Find A_8.

38. Find all positive integers n such that n + 9 and 16n + 9 and 27n + 9 are all perfect squares.

39. Find the number of positive integers which divide 10^{999} but not 10^{998}.

40. Let n = 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 .... 98 99100101. Compute the sum of the digits of n.

41. Find the number of positive integers x which satisfy the condition $\left[\frac{x}{99}\right]=\left[\frac{x}{101}\right]$.

42. Find all 7 digit numbers formed by using only the digits 5 and 7 and divisible by both 5 and 7.

43. Let n = 9 + 99 + 999 + ... + 999 .... 9, where the last summand consists of 1999 digits of 9. How many times will the digit 1 appear in n ?

44. Arrange the follwing numbers in the increasing order $\log_3 108$, $\log_4 192$, $\log_5 500$, $\log_6 1080$.

45. The function f defined on the set of ordered pairs of positive integers has the following properties.

(a) f(x, x) = x for all x.

(b) f(x,y) = f(y, x) for all x and y.

(c) (x + y) f(x, y) = yf(x, x + y) for all x and y.

Find f(52, 14)

46. Find a, b if $g(x) = \dfrac{ax+b}{7x-b}$ satisfies g(g(x)) = x for all $x \neq \dfrac{6}{7}$

47. Let f be a polynomial of degree 98 such that $f(k) = \dfrac{1}{k}$ for k = 1, 2, 3, ... 99. Determine f(100).

48. For any positive integer k, let $f_1(k)$ denote the square of the sum of the digits of k. For example $f_1(21) = (2 + 1)^2 = 9$. For $n \geq 2$, let $f_n(k) = f_1(f_{n-1}(k))$. Find $f_{1998}(11)$.

49. Given that $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

(a) Find the number of real roots of the quadratic equation $x^2 + 2|x| + 1 = 0$.

(b) Find the value of $\left[\frac{1}{4}\right]+\left[\frac{1}{4}+\frac{1}{50}\right]+\left[\frac{1}{4}+\frac{2}{50}\right]+...+\left[\frac{1}{4}+\frac{40}{50}\right]$ where the symbol [x] denotes the largest integer less than or equal to x.

50. The first digit from the left of a 4 digit number is equal to the number of zeros in the number. The second digit is equal to the number of digits 1, the third digit is equal to the number of digits 2 and the fourth digit is equal to the number of digits 3. How many numbers have this property.

51. Let A be the least number such that 10A is a perfect square and 35A is a perfect cube. Then the number of poistive divisors of A is

52. What is the least possible value of the expression 2008–BHA–SK–ARA if it is known that each alphabet represents a different non zero digit ?

53. The Fibonacci sequence is defined by $F_0 = 1$, $F_1 = 1$, $F_n = F_{n+1} + F_{n-2}$, $n \geq 2$

Show that $7F_{n+2}^3 - 7F_n^3 - F_{n+1}^3$ is divisible by F_{n+3}.

54. In the dog language BOW, the alphabet consists of the letters B, O, W only. Independently of the choice of the BOW word of length n ((i.e.) number of alphabets in the word is n) from which to start, one can construct all the BOW words with length n using iteratively the following rules.

(i) Reverse the order of the letters of the word (if BOWW is a word then if we reverse the order of letters we get WWOB)

(ii) replace two consecutive letters as follows :

BO $\rightarrow$ WW,	WW $\rightarrow$ BO,
WB $\rightarrow$ OO ,	OO $\rightarrow$ WB,
OW $\rightarrow$ BB,	BB $\rightarrow$ OW,

Given that

BBOWOBOWWOBOWWWOBOWWWWOBB

is a BOW word, does the BOW language have the following words ?

(a) BWOBWOBWOBWOBOWBOWBOWBOW B

(b) OBWOBWOBWOBWOWBOWBOWBOWBO

55. Show that the numbers 1 to 15 cannot be divided into a group A of 2 numbers and a group B of 13 numbers in such a way that the sum of the numbers in B is equal to the product of the numbers in A.

56. There are 13 white, 15 block and 17 red beads on a table. You have many number of beads of these colours with you. In one step 2 beads on the table of different colours are chosen by you and you replace each one by a bead of third colour from you. After how many such steps you will have all the beads of the same colour?

57. A 4-digit number n not containing the digt 9 is a square of an integer. If we increase every digit of n by 1 we get a square of another integer again. Find all such n.

58. (a) 28 integers are chosen from the interval [104, 208]. Show that there exist two of them with a common prime divisor.

(b) C is a point on the line segment AB. ACPQ and CBRS are squares drawn on the same side of AB. Prove that S is the orthocenter of the triangle APB.

59. The arithmetic mean of a number of pairwise distinct primes is 27. Determine the biggest prime among them.

60. Sixty five bugs are placed at different squares of a 9×9 board. A bug in each square moves to a horizontal or vertical adjacent square. No bug makes two horizontal or two vertical moves in succession. Show that after some moves, there will be at least two bugs in the same square.

61. Find the greatest common divisor of the numbers $n^n - n$, n = 3, 5, 7, 9, ...

62. (a) A book contained problems an Algebra, Geometry and Number theory. Mahadevan solved some of them. After checking the answers, he found that he answered correctly 50% probelms in Algebra, 70% in Geometry and 80% in Number theory. He further found that he solved correctly 62% of problems in Algebra and Number theory put together, 74% questions in Geometry and Number theory, 74% questions in Geometry and Number theory altogether. What is the percentage of correctly answered questions in all the three subjects?

(b) Find all pairs of positive integers (a, b) such that $a^b - b^a = 3$.

63. (a) Show that among any n + 1 whole numbers, one can find two numbers such that their difference is divisible by n.

(b) Show that for any natural number n, there is a positive integer all of whose digits are 5 or 0 and is divisible by n.

ANSWERS

1. 3,2 **5.** 2195 **6.** 1 value : 6 **7.** 1999 **8.** a = 2, b = – c **9.** odd number

10. {2,3,5,7,23,37,53,73,373} **11.** 16,8,4,2 **13.** 7 **15.** 16 **19.** 998001

21. No solution for $n \geq 5$, $n = 4$ **24.** 12, 16 or 4, 48

25. 1, 3, 11, 61, 3×11, 11×61, 3×61, $3 \times 11 \times 61$,

3^2, 11^2, 61^2, $3^2 \times 11$, $3^2 \times 61$, $3^2 \times 11 \times 61$,

$11^2 \times 3$, $11^2 \times 61$, $11^2 \times 61 \times 3$, $61^2 \times 3$,

$61^2 \times 11$, $61^2 \times 11^2 \times 3$, $3^2 \times 11^2$; $3^2 \times 11^2 \times 61$,

$11^2 \times 61^2$, $11^2 \times 61^2 \times 3$, $61^2 \times 3^2$, $61^2 \times 3^2 \times 11$,

$3^2 \times 11^2 \times 61^2$.

27 Solution for x & the correspodning values for y.

26. 142857 **27.** The highest power of 3 dividing N is 3 **28.** (4, 2)

29. 7744 **30.** 661 **31.** 1024 **32.** 1 **33.** 142857 **34.** 2×3^5

35. 192 **36.** 2200 **37.** 9 **38.** Only one integer n = 280

39. 1999 **40.** 903 **41.** 2499 number of such integers are possible

42. [7775775, 7757575, 5577775, 7575575, 5777555, 7755755, 5755575, 5557755, 7555555]

43. 1998 **44.** $\log_4 192 < \log_5 500 < \log_6 1080 < \log_3 108$

45. 364 **46.** a = 6, b is any real number other than –36/7

47. f(100) = 1/50 **48.** $f_{1998}(11) = 169$ **49.** (a) No real roots (b) 3

50. 2 **51.** 72 **52.** 106

56. We cann't end up with all beadds of the same color.

57. The value of n = 2025 **59.** 139 **61.** 24 **62.** (a) 65%, (b) a = 4, b = 1

CHAPTER 4 Inequalities

1. Four bags were to be weighed but the scale could only weigh weights in excess of 100kg. If the bags were weighed in pairs and the weights were found to be 103, 105, 106, 107 and 109, find the weight of the lightest bag.

2. If $0 < a < 1$, prove that the value of the expression $\left\{\frac{\sqrt{1+a}}{\sqrt{1+a}-\sqrt{1-a}}+\frac{1-a}{\sqrt{1-a^2}-1+a}\times\sqrt{\frac{1}{a^2}-1}-\frac{1}{a}\right\}$ lies between –2 and +2.

3. The cost of 175 chocolates is more than that of 125 cups of coffee but is less than that of 126 cups of coffee. Is a rupee enough of buy a cup of coffee and 3 chocolates if each cost in whole paise ? Justify your answer.

4. If a, b, c are the sides of a triangle, prove the following inequality :

$$\frac{a}{c+a-b}+\frac{b}{a+b-c}+\frac{c}{b+c-a}\geq 3.$$

5. Let a, b, c be real numbers such that $a + b + c = 1$. Prove that $a^2 + b^2 + c^2 \geq 4(ab + bc + ca) - 1$. When does the equality hold ?

6. Let a, b, c be positive real numbers such that $a + b + c \geq abc$. Prove that $a^2 + b^2 + c^2 \geq \sqrt{3}\,abc$.

7. If x, y, z are the sides of a triangle, prove that $|x^2(y - z) + y^2(z - x) + z^2(x - y)| < xyz$.

8. Let x, y be positive real such that, $x + y = 2$. Prove that $x^3y^3(x^3 + y^3) \leq 2$.

9. Let a and b be positive numbers. Prove that $\sqrt[3]{\frac{a}{b}}+\sqrt[3]{\frac{b}{a}}\leq\sqrt[3]{2(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}$.

10. Show that $\log_4 192 < \log_5 500 < \log_6 1080 < \log_3 108$.

11. If a, b, c are three real numbers such that $|a - b| \geq |c|$, $|b - c| \geq |a|$, $|c - a| \geq |b|$ prove that one of a, b, c is the sum of the other two.

12. Let a, b, c be three positive real numbers such that $a + b + c = 1$. Let $\lambda = \min\{a^3 + a^2bc, b^3 + ab^2c, c^3 + abc^2\}$. Prove that the roots of the equation $x^2 + x + 4\lambda = 0$ are real.

13. Given that a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3abc$. Prove that,

$$\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2}\geq\frac{9}{a+b+c}.$$

14. Given that a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3abc$. Prove that

$$\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2} \geq \frac{a}{a+b+c}.$$

15. Find the maximum positive real number k such that $\frac{xy}{\sqrt{(x^2+y^2)(3x^2+y^2)}} \leq \frac{1}{k}$ for all positive real numbers x and y.

16. Let a, b, c, d, be positive real numbers. Prove that $\frac{1}{a}+\frac{1}{b}+\frac{4}{c}+\frac{16}{d} \geq \frac{64}{a+b+c+d}$

17. If a, b, c, d are four real numbers such that $a + 2b + 3c + 4d \geq 30$, prove that $a^2 + b^2 + c^2 + d^2 \geq 30$.

18. The number of pairs of prime numbers (p,q) satisfying the condition $\frac{51}{100} < \frac{1}{p}+\frac{1}{q} < \frac{5}{6}$ will be

19. (a) If P_1, P_2, ...P_{2014} is an arbitrary rerrangement of 1, 2, 3,..........2014. Show that

$$\frac{1}{P_1+P_2}+\frac{1}{P_2+P_3}+.........+\frac{1}{P_{2013}+P_{2014}} > \frac{2013}{2016}$$

(b) Find positive integers n such that $\sqrt{n-1}+\sqrt{n+1}$ is rational.

20. If x, y, z are each greater than 1, show that $\frac{x^4}{(y-1)^2}+\frac{y^4}{(z-1)^2}+\frac{z^4}{(x-1)} \geq 48$

21. Let a, b, c, d be positive real numbers. Show that

$$\frac{ab+bc+ca}{a^3+b^3+c^3}+\frac{ab+bd+da}{a^3+b^3+d^3}+\frac{ac+cd+da}{a^3+c^3+d^3}+\frac{bc+cd+db}{b^3+c^3+d^3}$$

$$\leq \min\left\{\frac{a^2+b^2}{(ab)^{3/2}}+\frac{c^2+d^2}{(cd)^{3/2}}+\frac{a^2+c^2}{(ac)^{3/2}}+\frac{b^2+d^2}{(bd)^{3/2}}+\frac{a^2+d^2}{(ad)^{3/2}}+\frac{b^2+c^2}{(bc)^{3/2}}\right\}$$

22. If a, b, c, d are positive real numbers such that $a + b + c + d = 1$,

show that $\frac{a^3}{b+c}+\frac{b^3}{c+d}+\frac{c^3}{d+a}+\frac{d^3}{a+b} \geq \frac{1}{8}$.

23. a, b, c are real numbers such that $a + b + c = 0$ and $a^2 + b^2 + c^2 = 1$. Prove that $a^2b^2c^2 \leq \frac{1}{54}$. When does the equality hold?

ANSWERS

1. 51 kg **3.** It is not enough to buy one cup of coffee & 3 chocolates in Re. 1. **5.** a = b = c = 1/3

15. $K_{max} = \sqrt{3}+1$ **18.** 49